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MATHEMATICAL INFINITY AND THE DIFFERENTIAL.

By FRANKLIN A. BECHER, Milwaukee, Wisconsin.

Mathematics, as defined by the great mathematician, Benjamin Pierce, is the science which draws necessary conclusions. In its broadest sense, it deals with conceptions from which necessary conclusions are drawn. A mathematical conception is any conception which, by means of a finite number of specified elements, is precisely and completely defined and determined. To denote the dependence of a mathematical conception on its elements, the word "manifoldness," introduced by Riemann, has been recently adopted. Manifoldness may be looked upon as the genus, and function, as the species. This conception reaches down to the very foundation of mathematical concepts and principles. It is the central idea from which the whole field and range of the mathematical sciences may be surveyed. Time, space, and numbers are included in the notion, manifoldness.

Manifoldness may be defined according to Dr. Cantor as being in general every *muchness* or complexity which may be conceived as a unit, or a number of objects, conceptions, or elements which are united in one law or system.

Manifoldness may be devided into discrete and continuous. Proceeding with the conception of whole numbers as it is obtained by counting and extending the same by means of the divisibility of numbers so as to include the conception of the rational system of numbers, we have one of the elements which enter into the conception of a discrete manifoldness. The irrational system of numbers is included in the conception of continuous manifoldness. This must not be considered as an inherent division, for it is well to note here that in the higher analysis, in one instance and for one purpose, a conception may be considered as a discrete manifoldness and for another purpose as a continuous manifoldness.

The three laws of operation, i. e., the law of commutation, of association, and of distribution, hold good in all forms of calculation, whether discrets or continuous manifoldness. From these laws, the four processes, addition, subtraction, multiplication, and division are derived.

Number, in all its forms, whether finite or infinite, rational or irrational, constant or variable, continuous or discontinuous, is included as one of the elements of manifoldness.

We will now consider number with special reference to its limits, infinity and zero, by the introduction of the conception of variability, of continuity, and of the differential.

By means of an unlimited continuous series of rational numbers, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots\ldots \alpha_n$, whose terms have the property that there be given to every number δ, however small, a place n, from which the difference of all succeeding numbers remains smaller than δ, we define a definite number which is called a limit of this series. The creation of this conception admits of a comparison of rational numbers with respect to their magnitude. If all the numbers of the series differ after the place α_n by less than the number δ, then the limit is a number which lies between $\alpha_n - \delta$ and $\alpha_n + \delta$, which, because δ may be chosen as small as we please, can be expressed by a rational number as near as we please.

The totality of all numbers of an interval, for example, 0 to 1, consists not only of all numbers between 0 and 1, but of the totality of all numbers which may be interpolated between the limiting values of the defined series of numbers. This totality we dominate the aggregate or inclusive of the continuous series of numbers.

It is apparent that the conception of a limit of variability and of continuity have their root in irrationality. The two conceptions attached to a limit are in their nature entirely different. In the first instance, a limit may be defined as a limit of a variable, a limitless increasing or decreasing; in the second instance a limit means that which exceeds all limits of measurable number, either because it possesses no magnitude or because the amount or extent would not be exhausted by means of all the series of all numbers though they were being perfected. In the first case, we deal with variable numbers ; in the second case with the conception of the absolute value of the numbers derived from the formation of zero and the conception of infinity.

Zero and infinity are the limits of the natural series of numbers. They are derived in the same manner as the rational series of numbers. Infinity is the result of unlimited addition of unity or other positive numbers, the unlimited multiplication of whole numbers except unity. Zero is derived from the subtraction of two equal numbers. These are the fundamental conceptions of zero and infinity as derived in the lower analysis. It is evident from the different ways in which each of these symbols are derived that they have different meanings attached to them. We may note here that every problem carries inherently with it its solution. The meaning of every symbol depends upon its origin, deriv-

ation and relation. In different problems they may have different meanings. Symbols of quantity, like words, have different definitions, and these are to be determined according to the nature of the problem and their relation to other symbols.

In the higher analysis, the conceptions of infinity and zero present themselves more systematically in the developement of infinite series, infinite products, infinite continued fractions, etc. An infinite number is defined as a variable number, whose absolute value is conceived as being in an unlimited state of increasing or decreasing. In the first instance it is called infinitely large, in the second, infinitely small. The addition of a number of infinitely large or infinitely small numbers will produce an infinitely large or small number. The difference between two infinitely large or infinitely small numbers, where either or both are equal, is zero. However, if they are not equal, the difference can never be a finite number, but must always be an infinite number; otherwise an infinite number would be increased or decreased by a finite number, which is without meaning.

The addition and subtraction of infinite numbers can never produce anything else than infinite numbers or, in a particular case, zero. Again the multiplication or division of an infinite number by a finite number or by infinity will produce like results, i. e., it may be merely an indicated operation, not a completed operation. For instance,

$$2 \times \infty = 2\,\infty\,;\ n \times \infty = n\,\infty\,;\ \infty \times \infty = \infty^{2},\ \&c.$$

It is apparent that the unlimited number of changes which may be thought of under the conception of infinity as defined here are extraordinarily manifold.

If we conceive an infinite number to grow so that it is continuously twice as large as any other infinite number, then the first is derived from the second by multiplying by two or the second by dividing by two. Multiplying an infinite number by another gives us infinity of a higher power or dividing gives us infinity of a lower power. Every change in value of a variable suggests an increment.

There are two kinds of conceptions associated with increments: the one is that the absolute value of the increment is capable of divisibility. The conditions, however, of which are such that it cannot be conceived smaller. The other is that the absolute value is incapable of divisibility. In the first instance the increments are of such a nature that the variables must stand in a certain relation to one another and if this takes place they are known in higher mathematics as differentials; those of the second kind are of that nature that they do not stand in any relation to one another; these may be called absolute elements of quantity.

Thus, if we pass from one interval of value of a variable to another, there lies between the two a difference which must be considered as possessing quantity, but does not possess the capability of divisibility and this difference in in-

crements is an element of quantity. Increments and differentials are not identical. The former are vested with quantity, while the latter are vested with quality, i. e., they are formal in their nature.

Milwaukee, Wisconsin, September, 1896.

A PROBLEM IN ASTRONOMY.

By G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

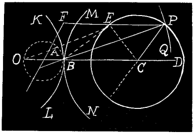
To find the Distance from the Earth to the Sun knowing the distance from an Inferior planet to the Sun supposing the planets to describe circles around the Sun.

Let P be a point on the epicyclic curve PQ, OC the radius of the deferent, CP the radius of the epicycle. Let $CO : CB = n : 1$. $\therefore CB = \frac{CO}{n}$.

Then $BO = CO - CB = CO - \frac{CO}{n} = CO(1 - \frac{1}{n})$.

Now the angular velocity$=\dfrac{\text{transverse velocity}}{\text{radius vector}}$.

The transverse velocity of E is represented by EA in magnitude, and in direction by BA. Let the linear velocity of the mean point (C) be V, the linear velocity of the moving point in the epicycle is



$$nV.\frac{PC}{CO}.$$

$\therefore$ The transverse velocity=

$$\frac{nV}{CO}.EB\cos BEO;$$

$$\text{but } \cos BEO = \frac{(BE)^2 + (EO)^2 - (BO)^2}{2BE.EO}.$$

$\therefore$ Transverse velocity $= \frac{nV}{CO}.\frac{BE^2 + EO^2 - BO^2}{2EO}$. Also radius vector$=EO$.

$$\therefore \text{Angular velocity} = \frac{nV}{CO}.\frac{BE^2 + EO^2 - BO^2}{2EO^2} \ldots\ldots\ldots\ldots (A).$$

Let the deferential angle$=\theta$, then angle $ECD=(n-1)\theta$.

$$\therefore\ BE^2=CE^2+CB^2+2CE.CB\cos(n-1)\theta$$

$$=CE^2+\frac{CO^2}{n^2}+2CE.\frac{CO}{n}\cos(n-1)\theta$$

$$EO^2=CO^2+CE^2+2CO.CE\cos(n-1)\theta$$

$$BO^2=CO^2(1-\frac{1}{n})^2.$$

Substituting these values in (A) we get

$$\text{Angular velocity}=\frac{V}{CO}.\frac{CO^2+nCE^2+(n+1)CO.CE\cos(n-1)\theta}{CO^2+CE^2+2CO.CE\cos(n-1)\theta}.$$

Now in inferior conjunction, if the moving planet is inferior, $(n-1)\theta=180°$.

$$\therefore\ \text{Angular velocity}=\frac{V}{CO}.\frac{CO^2+nCE^2-(n+1)CO.CE}{CO^2+CE^2-2CO.CE}.$$

Let $CO=R$, $CE=r$.

$$\text{Then Angular velocity}=\frac{V}{R}.\frac{R^2+nr^2-(n+1)R.r}{(R-r)^2}.$$

Now $n=\left(\frac{R}{r}\right)^{\frac{3}{2}}$, also putting $\frac{V}{R}=\omega$.

$$\therefore\ \text{Angular velocity}=\omega.\frac{R^2+\left(\frac{R}{r}\right)^{\frac{3}{2}}r^2-\left\{\left(\frac{R}{r}\right)^{\frac{3}{2}}+1\right\}R.r}{(R-r)^2}$$

$$=\omega\frac{1+\left(\frac{R}{r}\right)^{\frac{3}{2}}\frac{r^2}{R^2}-\frac{R^{\frac{3}{2}}+r^{\frac{3}{2}}}{r^{\frac{3}{2}}}.\frac{r}{R}}{\left(1-\frac{r}{R}\right)^2}\ \ldots\ldots\ldots\ldots\ldots(B)$$

$$=\omega\frac{\frac{R^2}{r^2}+\left(\frac{R}{r}\right)^{\frac{3}{2}}-\left\{\left(\frac{R}{r}\right)^{\frac{3}{2}}+1\right\}\frac{R}{r}}{\left(\frac{R}{r}-1\right)^2}\ \ldots\ldots\ldots\ldots(C).$$

Let the distance from the earth to the sun be known to find the distance from the planet to the sun.

Let $\frac{r}{R}=\rho$, then (B) becomes

$$\text{Angular velocity}=\omega.\frac{1+\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}}-\rho}{(1-\rho)^2}=\omega\,.\,\frac{(1-\rho)-\rho^{-\frac{1}{2}}(1-\rho)}{(1-\rho)^2}$$

$$=\omega.\frac{1-\rho^{-\frac{1}{2}}}{1-\rho}=-\frac{\omega}{\sqrt{\rho}}.\frac{1-\rho^{\frac{1}{2}}}{1-\rho}=-\frac{\omega}{\sqrt{\rho}+\rho}\;\ldots\ldots\ldots\ldots(D).$$

Let the distance from the planet to the sun be known to find the distance from the earth to the sun.

Let $\frac{R}{r}=\rho'$, then (C) becomes

$$\text{Angular velocity}=\omega.\frac{\rho'^2+\rho'^{\frac{3}{2}}-\rho'^{\frac{5}{2}}-\rho'}{(\rho'-1)^2}=\omega\frac{\rho'(\rho'-1)-\rho'^{\frac{3}{2}}(\rho'-1)}{(\rho'-1)^2}$$

$$=\omega.\frac{\rho'-\rho'^{\frac{3}{2}}}{\rho'-1}=-\frac{\omega\rho'}{1+\sqrt{\rho'}}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(E).$$

Case I. A planet transits the sun's disc at such a rate that the sun's diameter S would be traversed in time t. Find the planet's distance from the sun.

Let ρ=planet's distance, unity being the earth's distance, and let ω be the earth's angular velocity around the sun=sun's angular velocity around the earth, and let t' be the time in which the sun in his annual course moves through a distance equal to his own apparent diameter; then $\omega t'=S$. From (D) the planet's angular velocity about the earth$=-\frac{\omega}{\sqrt{\rho}+\rho}$.

$\therefore$ That is the planet's retrograde gain on the sun is

$$\frac{\omega}{\sqrt{\rho}+\rho}+\omega=\frac{S}{t}=\frac{\omega t'}{t}.$$

$$\therefore\;\rho+\rho^{\frac{1}{2}}=\frac{t}{t'-t},\quad \therefore\;\rho^{\frac{1}{2}}=\tfrac{1}{2}\Big(\pm\frac{\sqrt{3t+t'}}{\sqrt{t'-t}}-1\Big).$$

$$\therefore\;\rho=\tfrac{1}{2}\left(\frac{t'+t}{t'-t}-\sqrt{\frac{3t+t'}{t'-t}}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Case II. If we wish to find the earth's distance knowing the planet's distance, then let the planet's distance be unity and the earth's distance$=\rho'$.

Proceeding the same as before using (E) we get

$$\frac{\omega\,\rho'}{1+\sqrt{\rho'}}+\omega=\frac{S}{t}=\frac{\omega t'}{t}.\quad\therefore\ \rho'-\frac{t'-t}{t}\sqrt{\rho'}=\frac{t'-t}{t};$$

$$\therefore\ \sqrt{\rho'}=\frac{1}{2t}\left\{(t'-t)+\sqrt{t'^2-3t^2+2tt'}\right\}.$$

$$\therefore\ \rho'=\frac{t'-t}{2t^2}\left\{(t'+t)+\sqrt{t'^2-3t^2+2tt'}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Suppose Venus transits the sun's disc at such a rate that the sun's apparent diameter would be traversed in $7\frac{1}{3}$ hours, and at the same time the sun in his annual course moves through a distance equal to his own apparent diameter in 12 hours. Required (1) the distance from Venus to the sun, the earth's distance being unity, and (2) the distance from the earth to the sun, Venus's distance being unity.

Now $t=7\frac{1}{3}$, $t'=12$; hence for first case substituting in (1)

$$\rho=\tfrac{1}{2}\left(\tfrac{29}{7}-\sqrt{\tfrac{51}{7}}\right)=.721824.$$

For the second case substitute in (2)

$$\rho'=\tfrac{7}{484}\{58+\sqrt{1428}\}=1.38538.$$

(The above is suggested in Proctor's Geometry of the Cycloid.)

A PROPOSITION IN DETERMINANTS.

By ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi.

THEOREM.—The product of two numbers, each the sum of four squares, is the sum of eight squares.

$$\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} \\ c+d\sqrt{-1} & a-b\sqrt{-1} \end{vmatrix}\times\begin{vmatrix} \alpha+\beta\sqrt{-1} & -\gamma+\delta\sqrt{-1} \\ \gamma+\delta\sqrt{-1} & \alpha-\beta\sqrt{-1} \end{vmatrix}$$

$$=\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} & 0 \\ c+d\sqrt{-1} & a-b\sqrt{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix}\times(-1)\begin{vmatrix} \alpha+\beta\sqrt{-1} & 0 & -\gamma+\delta\sqrt{-1} \\ \gamma+\delta\sqrt{-1} & 0 & \alpha-\beta\sqrt{-1} \\ 0 & 1 & 0 \end{vmatrix}$$

$$=(-1)\begin{vmatrix} a\alpha-b\beta+(a\beta+b\alpha)\sqrt{-1} & a\gamma-b\delta+(a\delta+b\gamma)\sqrt{-1} & -c+d\sqrt{-1} \\ c\alpha-d\beta+(c\beta+d\alpha)\sqrt{-1} & c\gamma-d\delta+(c\delta+d\gamma)\sqrt{-1} & a-b\sqrt{-1} \\ -\gamma+\delta\sqrt{-1} & \alpha-\beta\sqrt{-1} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} c\alpha-d\beta+(c\beta+d\alpha)\sqrt{-1} & c\gamma-d\delta+(c\delta+d\gamma)\sqrt{-1} \\ -c\gamma+d\delta+(c\delta+d\gamma)\sqrt{-1} & c\alpha-d\beta-(c\beta+d\alpha)\sqrt{-1} \end{vmatrix}$$

$$+ \begin{vmatrix} a\alpha-b\beta+(a\beta+b\alpha)\sqrt{-1} & a\gamma-b\delta+(a\delta+b\gamma)\sqrt{-1} \\ -a\gamma+b\delta+(a\delta+b\gamma)\sqrt{-1} & a\alpha-b\beta-(a\beta+b\alpha)\sqrt{-1} \end{vmatrix}$$

or $(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)=(c\alpha-d\beta)^2+(c\beta+d\alpha)^2$
$+(c\gamma-d\delta)^2+(c\delta+d\gamma)^2+(a\alpha-b\beta)^2+(a\beta+b\alpha)^2+(a\gamma-b\delta)^2+(a\delta+b\gamma)^2.$

Euler's Theorem is an easy corollary of this, and *vice-versa.*

University of Mississippi, March, 1896.

A METHOD OF SOLVING QUADRATIC EQUATIONS.

By Prof. HENRY HEATON, M. Sc., Atlantic, Iowa.

Let it be required to solve the equation

$$ax^2+bx+c=0 \qquad\qquad (1).$$

Transposing the middle term we have

$$ax^2+c=-bx \qquad\qquad (2).$$

Squaring, $a^2x^4+2acx^2+c^2=b^2x^2$ .... (3).

Subtracting $4acx^2$, $a^2x^4-2acx^2=(b^2-4ac)x^2$ .... (4).

Extracting the square root, $ax^2-c=\pm(\sqrt{b^2-4ac})x$ .... (5).

Adding equation (2), $2a^2x^2=(-b\pm\sqrt{b^2-4ac})x$ .... (6).

Whence $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Let it be required to solve the equation $3x^2-2x=21$.

Transposing $2x$ to the second member and 21 to the first, the equation becomes

$$3x^2-21=2x \qquad\qquad (7).$$

Squaring, $9x^4-126x^2+441=4x^2$ ....(8).

Adding twice $126x^2$, $9x^4+126x^2+441=256x^2$..............................(9).

Extracting the square root, $3x^2+21=\pm16x$..............................(10).

Adding equation (7), $6x^2=18x$ or $-14x$.

$\therefore\ x=3$ or $-2\frac{1}{3}$.
Is this new?

[NOTE.—We do not remember of ever having seen this method. If any of our readers have seen it elsewhere, please let us know. Editor.]

ON THE DOCTRINE OF PARALLELS.

By Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

I desire to enter my protest against any assumption that parallel lines, extended to an infinite distance, do, or do not, intersect. The human mind cannot comprehend the infinite and, therefore, we cannot determine the question. We may use modes of reasoning involving infinite quantities, but we can rely upon the results *only so far as human experience shows that they are correct.* It is trne, that a mode of reasoning in such cases. which leads to a result found by human experience to be correct in a particular case, may generally be assumed to be correct in all cases. Without human experience, the proposition that if two objects are moving in the same line in the same direction at different velocities, the one in advance will move over an appreciable space while the other is moving over the space between them and, therefore, that the one can never overtake the other, could never have been successfully denied. I hold that this doctrine applies to much of the discussion of the present day, and some of the propositions I have been able to deny, and old propositions denied I have been able to affirm, because I knew that *human experience had settled the matter.*

Whether Euclid's reasoning was, or was not correct, I have never seen a case in which the result which he reached has not been found to be absolutely correct by human experience.

The quotation which Professor Lyle makes from Lotze (Vol. II. page 375) involves the arrogant assumption that the human mind is infinite in the scope of its reasoning power. Mathematicians, of all men, should not claim that a proposition involving the infinite, cannot be true, because we cannot *comprehend the possibility* of its being true.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

62. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A dealer buys milk at $m=5$ cents per quart, and sells it at $n=6$ cents per quart. How much water has he put with the milk, if his rate of profit is $p=60\%$?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas, and J. F. YOTHERS, Westerville, Ohio.

$m(1+p)$=price at which a quart of pure milk would sell at a profit of $p\%$.

$\frac{m(1+p)}{n}$=number of quarts at n cents sold for $m(1+p)$ cents.

$\therefore \frac{m(1+p)}{n}-1=\frac{m(1+p)-n}{n}$=amount of water added to each quart of milk. Let $m=5$, $n=6$, $p=.60$.

$\therefore \frac{m(1+p)-n}{n}=\frac{1}{3}$. $\therefore$ He adds one quart of water to 3 quarts of milk.

Also solved by *E. R. ROBBINS*, *P. S. BERG*, *F. R. HONEY*.

63. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pennsylvania.

I owe A \$100 due in 2 years, and \$200 due in 4 years; when will the payment of \$300 equitably discharge the debt, money being worth 6%?

I. Solution by P. S. BERG, Larimore, North Dakota; EDWARD R. ROBBINS, Lawrenceville, New Jersey; FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Present worth of \$100 for 2 years at 6%=\$89.28.
Present worth of \$200 for 4 years at 6%=\$161.29.
\$250.57=\$89.28+\$161.29=sum of present worths,
The time required for \$250.57 at 6% to amount to \$300 is the time sought.
Interest of \$250.57 for one year at 6%=\$15.0342.
\$300−\$250.57=\$49.43, interest for the whole term.
Hence time equals 49.43÷15.0342=3.2878 years.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas; J. F. YOTHERS, Westerville, Ohio.

The interest on \$100 for 2 years at 6%=\$12.
The interest on \$200 for 4 years at 6%=\$48.
The interest on \$300 for 1 year at 6%=\$18.
(\$12+\$48)÷\$18=\$60÷\$18=3 years, 4 months.

Or, \$100 for 2 years=\$ 200 for 1 year.
\$200 for 4 years=\$ 800 for 1 year.

\$1000 for 1 years.

\$1000÷\$300=3 years, 4 months.

PROBLEMS.

67. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A agreed to work a year for \$300 and a suit of clothes. At the end of five months he left, receiving for his wages \$60 and the clothes. What was the suit worth?

68. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The population of a city is annually increasing $m=2\frac{1}{2}\%$. If the population now is $n=3$ years ago? At this rate of increase, what will the population

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

58. Proposed by I. J. SCHWATT, Ph. D., Instructor in Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

1. The point of intersection K_a' of the tangent drawn to the circumcircle about the triangle ABC at A and the side BC is harmonic conjugate to K_a with respect to BC. (K_a is the point where the symmedian line through A of the triangle ABC meets the side BC.)

2. The point K_a' is the center of the Apollonius circle passing through A of the triangle ABC.

3. Grebes point is on the line joining the middle point of any side of a triangle with the middle point of the altitude to this side.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

1. In trilinears, the equation to the circumcircle of the triangle of reference is

$$a\beta\gamma+b\alpha\gamma+c\alpha\beta=0\ldots\ldots\ldots\ldots(1).$$

The tangent to this circle at A is

$$b\gamma+c\beta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The equation to the symmedian through A is

$$b\gamma-c\beta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

(2) and (3) are conjugates to $\beta=0$, $\gamma=0$.

2. The circle of Apollonius passes through A and the points of intersection of the internal and external bisectors of the angle A of the triangle of reference with the side BC. The coordinates of the center of this circle are plainly the half sum of those of the intersections of the bisectors with BC, or $\left(0, \dfrac{ab^2\sin C}{b^2-c^2}, \dfrac{ac^2\sin B}{b^2-c^2}\right)$. This is the point of intersection of (2) and $\alpha=0$.

3. The coordinates of Grebe's point are proportional to a, b, c.

The mid-point of a is $(0, \frac{a}{2}\sin C, \frac{a}{2}\sin B)$, and of the altitude on a, $\left(\dfrac{\triangle}{a}, \dfrac{b}{4}\sin 2C, \dfrac{c}{4}\sin 2B\right)$.

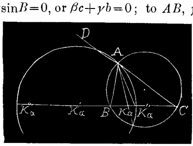
The line through these points is $\alpha\sin(B-C)+\gamma\sin C-\beta\sin B=0$, which is satisfied by $\alpha=a$, $\beta=b$, $\gamma=c$.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

Using trilinear coordinates we get,

(1), equation to AK_a' is, $\beta\sin C+\gamma\sin B=0$, or $\beta c+\gamma b=0$; to AB, $\gamma=0$; to AK_a, $\beta c-\gamma b=0$; to AC, $\beta=0$.

$\therefore AK_a'$, AB, AK_a, AC form a harmonic pencil.



$\therefore K_a'$ and K_a are harmonic conjugate.

(2). Draw AK_a''' bisecting $\angle DAB$.

Then $\angle AK_a'B=\angle K_a'K_a'''A+\angle K_a'AK_a'''=B-C$, and

$\angle K_a'AK_a'''=\angle K_a'''AB-\angle K_a'AB=90^\circ-\frac{1}{2}A-C=\frac{1}{2}(B-C)$.

$\therefore \angle K_a'K_a'''A=\angle K_a'AK_a'''$; $\therefore K_a'K_a'''=AK_a'=AK_a''$.

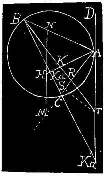
$\therefore K_a'$ is the center of the required circle.

(3). The equation to the straight line through Grebe's point and the mid-point of BC is the same as the equation of the straight line through mid-point of BC and the midpoint of the perpendicular from A on BC, both being $\sin(B-C)\alpha-\sin B\beta+\sin C\gamma=0$.

III. **Solution by F. M. McGAW, A. M., Professor of Mathematics, Bordentown Military Institute, Bordentown, New Jersey, and J. C. CORBIN, Pine Bluff, Arkansas.**

(1). Let ABC be $\triangle$; AK_a', the tangent at A; and K_a point where sym-

median meets the side BC. Take $AN=AC$, and $AM=AB$. Then in $\triangle ANM$, AH, the median, is the symmedian of $\triangle ABC$. Lines BC, and MN are antiparallel. Also, since $\measuredangle ABC=\measuredangle BAD$, each being measured by $\frac{1}{2}$arc AC, the lines BC and AK_a' are antiparallel. Wherefore MN is parallel to the tangent line AK_a'.



Now we have a pencil of four rays AB, AH, AC, AK_a' in which one ray AH bisects a line parallel to its conjugate, and included between the other pair of conjugate rays; hence the pencil is harmonic, and any line, as BK_a', drawn across the pencil will cut out an harmonic range $\{BC, K_aK_a'\}$. Q. E. D.

(3). Draw ST perpendicular to AC at its middle point S; draw BT, and it is a symmedian line (Halsted: Syn. Geom. §648.), hence it passes through Grebe's point (or Lemoine's Point) K. Now as $A\{BC, K_aK_a'\}$ is an harmonic pencil, $\{BR, KT\}$ is an harmonic range; whence $S\{BR, KT\}$ is an harmonic pencil. Draw altitude BP, and it is $\parallel$ to ray ST, and is therefore bisected by the ray SK, the conjugate of ST. Therefore the line joining the middle point, S, of a side, and the middle point of the altitude to that side passes through Grebe's point.

Q. E. D.

59. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that the tangent plane at any point of the surface $a^2x^2+b^2y^2+c^2z^2=2bcyz+2acxz+2abxy$ intersects the surface $ayz+bzx+cxy=0$ in two straight lines at right angles to one another.

Solution by the PROPOSER.

The tangent plane at (x', y', z') to $F=0$..........................(1) is

$$(x-x')\frac{dF}{dx'}+(y-y')\frac{dF}{dy'}+(z-z')\frac{dF}{dz'}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Here $F=a^2x^2+b^2y^2+c^2z^2-2bcyz-2acxz-2abxy$..................(3).

$$\frac{dF}{dx'}=2a(ax'-by'-cz'),\quad \frac{dF}{dy'}=2b(-ax'+by'-cz'),$$

$$\frac{dF}{dz'}=2c(-ax'-by'+cz') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Then (2) becomes by aid of (3),

$$a(ax'-by'-cz')x+b(-ax'+by'-cz')y+c(-ax'-by'+cz')z=0 \quad \ldots\ldots\ldots\ldots (5).$$

It may be shown that the condition that

$lx+my+nz=0$ ....(6) cuts $ayz+bxz+cxy=0$....... ... .(7)

in two straight lines including a right angle is $amn+bnl+clm=0$...........(8).

Comparing (5) and (6), $l=a(ax'-by'-cz')$, $m=b(-ax'+by'-cz')$, $n=c(-ax'-by'+cz')$, and (8) becomes

$abc\{a^2x'^2-(by'-cz')^2+b^2y'^2-(cz'-ax')^2+c^2z'^2-(ax'-by')^2\}=0$ ....(9),

an identity by aid of (3).

Also solved by *HENRY HEATON* and *J. SCHEFFER*.

PROBLEMS.

65. Proposed by I. J. SCHWATT, Ph. D., Professor of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania.

Prove in a pure geometrical way the following:

The axes of the ellipse isogonal to Lemoine's line with respect to a triangle (Steiner's ellipse) are parallel to Simson's lines belonging to the extremities of Brocard's Diameter.

66. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The locus of points whose polars with respect to a given parabola touch the circle of curvature at the vertex is an equilateral hyperbola.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

32. Proposed by OTTO CLAYTON, A. B., Fowler, Indiana.

The wheel of a wind pump has 60 fans, each turned at an angle of 45° to the direction of the axis, and each having 150 square inches exposed to the wind. If the wind blows with a velocity of V and the wheel rotates with velocity ω, what is the component of force or pressure along the axis if it is turned at an angte α to the direction of the wind assuming the resistance of the wheel in turning to be R?

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A=projecting area of fans exposed to the wind, in square feet,
V=velocity of wind in feet per second,
H=horse hower of pump,
R=extreme radius of fans in feet,

r—inner radius of fans in feet,

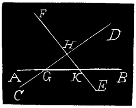
$l=\sqrt{\dfrac{R^2+r^2}{2}}$, = radius of center of percussion, in feet,

n—number of revolutions of fans per minute,

β—mean angle of fans to the plane of motion.

By Nystrom's Mechanics we get $H=\dfrac{Aln\sin\beta\cos\beta}{1.540000}\left(V-\dfrac{2ln\sin\beta}{19}\right)^2$.

Let AB, CD, FE be the direction of the axis, fans, and wind, respectively.



$\angle HKG=\alpha$, $\angle KGH=\dfrac{\pi}{4}$. $\therefore$ $\angle GKH=\left(\dfrac{3\pi}{4}-\alpha\right)$.

Then $A=60\times150\times\sin\left(\dfrac{3\pi}{4}-\alpha\right)$

$\div144=\frac{125}{2}\sin\left(\dfrac{3\pi}{4}-\alpha\right)$.

$n=\omega$, $\beta=\dfrac{\pi}{4}$. $\therefore$ $H=\dfrac{125l\omega\sin\left(\dfrac{3\pi}{4}-\alpha\right)\left(19V-l\omega\sqrt{2}\right)^2}{117040000}$.

$H'=H-R/33000$—effective horse power.

35. Proposed by **G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.**

A man weighs 150 pounds; his balloon with all its attachments weighs 500 pounds. What volume of pure hydrogen must be made and put into the balloon so that it will be on the point of ascending with the man? How many kilograms of zinc and of hydrogen sulphate will be used generating the hydrogen? Give volume of hydrogen in cubic feet, given that one litre of hydrogen weighs .0896 grams.

Solution by **P. S. BERG, Larimore, North Dakota, and the PROPOSER.**

1 grain = .0022046 pounds. 1 cubic foot = 28.315 litres.

Let temperature and pressure be normal.

$\therefore$ 1 cubic foot of hydrogen weighs $28.315\times.0896\times.0022046=.005593123$ pounds.

1 cubic foot of air weighs $28.315\times1.293\times.0022046=.080713261$ pounds.

$\therefore$ The lifting power of 1 cubic foot of hydrogen is .080713261 pounds $-$ .005593123 pounds = .075120138 pounds.

500 pounds + 150 pounds = 650 pounds.

$650\div.075120138=8652.806$ cubic feet of hydrogen.

$8652.806\times.005593123\div2.2046=21.9524$ kilograms of hydrogen used.

$Zn+H_2SO_4=ZnSO_4+H_2$. $\therefore$ $H_2SO_4:H_2=x:21.9524$.

$98:2=x:21.9524$. $\therefore$ $x=1075.6676$ kilograms of hydrogen sulphate.

$Zn:H_2=x:21.9524$. $65:2=x:21.9524$. $\therefore$ $x=713.453$ kilograms of zinc.

Also solved by *A. P. REED.*

PROBLEMS.

44. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

There is a triangle whose sides repulse a center of force within the triangle with an intensity that varies inversely as the distance of the center of force from each point of the sides of the triangle. What is the position of equilibrium of the center?

45. Proposed by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

A fifty-pound cannon-ball is projected vertically upward with a velocity of 300 feet per second. Find the height to which it will rise and the time of flight, assuming the initial resistance of the air on the ball to be 10 pounds and the resistance to vary as the square of the velocity.

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

64. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

A man raises 1 chicken the first year; 6, the second; 35, the third; 180, the fourth; 921, the fifth; 4626, the sixth; 23215, the seventh; 116160, the eighth; and so on. How many does he raise the 20th year, and how many in the twenty years?

I. Solution by A. H. HOLMES, Box 963, Brunswick, Maine.

We easily find by inspection $U_{x+1}-5U_x=\dfrac{4^{\frac{x+1}{2}}-1}{3}$, or $\dfrac{4^{\frac{x+2}{2}}-1}{3}$, according as x is odd or even. Integrating and reducing, we have

$$U_x=\tfrac{1}{4}\left[5^x+4\times5^{x-2}+4^2\times5^{x-4}+\text{etc.}-\frac{4^{\frac{x+1}{2}}-1}{3}\text{ or }\frac{4^{\frac{x+2}{2}}-1}{3}\right].$$

$$\text{Summing, } S_x=\tfrac{1}{16}\left[5^{x+1}+4\times5^{x-1}+4^2\times5^{x-3}+\text{etc.}-\frac{23\times4^{\frac{x+2}{2}}-12x-47}{9},\right.$$

$$\left.\text{or }\frac{11\times4^{\frac{x+3}{2}}-12x-47}{9}\right].$$

Putting $x=20$, and performing operations indicated, we have,
$U_{20}=28,383,163,779,300$, and $S_{20}=35,478,954,491,110$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

The numbers in the problem may be represented under the following form :

1	6	35	180	921	4626
$5\times0+1$,	$5\times1+1$,	$5\times6+5$,	$5\times35+5$,	$5\times180+21$,	$5\times921+21$,

23215	116160	
$5\times4626+85$,	$5\times23215+85$,	etc.

The general term of the numbers 1, 5, 21, 82, etc., is $\frac{1}{3}(4^x-1)$, as can be easily found by Finite Differences. Expressing the $(2x-1)$th term of the above series by $F(2x-1)$, we have, by Finite Differences, $F(2x-1)=C.5^{2x-1}+C_1.4^x+C_2$. Substituting for x successively 1, 2, 3, we have the three equations: $5C+4C_1+C_2=1$, $125C+16C_1+C_2=35$, $3125C+64C_1+C_2=921$, whence $C=25/84$, $C_1=-1/7$, $C_2=1/12$.

$$\therefore\ F(2x-1)=\frac{5^{2x+1}}{84}-\frac{4^x}{7}+\tfrac{1}{12}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{I}).$$

To find $F(2x)$, multiply $F(2x-1)$ by 5 and add $\frac{1}{3}(4^x-1)$, thus,

$$F(2x)=\frac{5^{2x+2}}{84}-\tfrac{8}{21}.4x+\tfrac{1}{12}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{II}).$$

By summing the geometrical series $5^3+5^5+\ldots\ldots\ 5^{2x-1}$, $5^4+5^6+\ldots\ldots\ 5^{2x+2}$, $4+4^2+4^3+\ldots\ldots4^x$, we find

$$\sum F(2x-1)=\frac{5^{2x+3}}{2016}-\frac{4^{x+1}}{21}+\tfrac{1}{12}x+\tfrac{259}{2016},\ \text{and}$$

$$\sum F(2x)=\frac{5^{2x+4}}{2016}-\tfrac{8}{63}.4^{x+1}+\tfrac{1}{12}x+\tfrac{399}{2016}.$$

Consequently $$\sum_{x-1}^{x-2n-1}(x)=\frac{5^{2n+2}}{336}-\tfrac{5}{63}.4^{n+1}+\tfrac{1}{6}n+\tfrac{35}{144}\ \ldots\ldots\ldots\ldots(\mathrm{III});$$

$$\sum_{x-1}^{x-2n}(x)=\frac{5^{2n+3}}{336}-\tfrac{11}{63}.4^{n+1}+\tfrac{1}{6}n+\tfrac{47}{144}\ \ldots\ldots\ldots\ldots(\mathrm{IV}).$$

The formulae I and III are to be employed for an odd number of terms, and II and IV for an even one. Thus, $F(20)=\frac{5^{22}}{84}-\frac{8}{21}.4^{10}+\frac{1}{12}=28383163779300$;

$\sum F(20)=\frac{5^{23}}{336}-\frac{11}{63}.4^{11}+\frac{10}{6}+\frac{47}{144}=35478954491110.$

III. **Solution by A. M. HUGHLETT, A. M., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Virginia.**

1. Write out to "n" terms the series: $1, 5, 25, 125, 625, \ldots\ldots 5^{n+1}$.
2. Begin at 3rd term and write the series: $4, 20, 100, \ldots\ldots 4.5^{n-3}$.
3. Begin at 5th term and write the series: $16, 4^2.5, 4^2.5^{n-5}$.

....

4. Begin at $(n-1)$th term and write the series: $4^{\frac{n-1}{2}}, 4^{\frac{n-2}{2}}.5$, n being even.
5. Begin at nth term and write the series: $4^{\frac{n-1}{2}}$, n being odd.

The nth term in the required series is the sum of all the numbers in the nth term of the above arrangement plus all that precede it. Denote the sum by s, then, if n is even,

$$s=\left\{\begin{array}{l}1+5+25\ldots\ldots+5^{n-1}\\ +4+4.5\ldots\ldots+4.5^{n-3}\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ +\qquad 4^{\frac{n-2}{2}}+4^{\frac{n-1}{2}}.5\end{array}\right\}=\frac{5^n-1}{4}+\frac{4(5^{n-2}-1)}{4}\ldots\ldots\ldots\frac{4^{\frac{n-2}{2}}(5^2-1)}{4}\quad(1).$$

If n is odd,

$$s=\left\{\begin{array}{l}1+5+35\ldots\ldots\ldots 5^{n-1}\\ 4+4.5\ldots\ldots\ldots 4.5^{n-3}\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ +\qquad 4^{\frac{n-1}{2}}\end{array}\right\}=\frac{5^n-1}{4}+\frac{4(5^{n-2}-1)}{4}\ldots\ldots\ldots 4^{\frac{n-1}{2}}\frac{(5-1)}{4}\quad(2).$$

(1) finally becomes: $\tfrac{1}{84}\{5^{n+2}+7-8.4^{\frac{n+2}{2}}\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$,

and (2) becomes $\tfrac{1}{84}\{5^{n+2}+7-12.4^{\frac{n+1}{2}}\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$.

(3) gives the even terms; (4) gives the odd terms. To sum the series:—

1st term by $(4)=\tfrac{1}{84}(5^3+7-12.4)$
2nd term by $(3)=\tfrac{1}{84}(5^4+7-8.4^2)$
3rd term by $(4)=\tfrac{1}{84}(5^5+7-12.4^2)$
4th term by $(3)=\tfrac{1}{84}(5^6+7-8.4^3)$
5th term by $(4)=\tfrac{1}{84}(5^7+7-12.4^3)$

....

nth term by $(3)=\tfrac{1}{84}(5^{n+2}+7-8.4^{\frac{n+2}{2}})$, n being even,

nth term by $(4)=\tfrac{1}{84}(5^{n+2}+7-12.4^{\frac{n+1}{2}})$, n being odd.

Denote the sum by S, then, *n being even*,

$$S = \tfrac{1}{84}\left\{\frac{5^{n+3}-125}{4}+7n-11\frac{(4^{\frac{n+4}{2}}-16)}{3}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Similarly, n being odd,

$$S = \tfrac{1}{84}\left\{\frac{5^{n+3}-125}{4}+7n-\frac{20.4^{\frac{n+3}{2}}-176}{3}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

By (3), the 20th term is : $\tfrac{1}{84}\{5^{22}+7-8.4^{11}\}=28{,}383{,}163{,}779{,}300.$

By (5), the twenty terms are : $\tfrac{1}{84}\{\frac{5^{23}-125}{4}+140-11\,\frac{(4^{12}-16)}{3}\}$

$=35{,}478{,}954{,}491{,}110.$

IV. Solution by the PROPOSER.

Let it be required to sum to n terms and find the nth term of the series :

$1+6x+35x^2+180x^3+921x^4+4626x^5+23215x^6+116160x^7+\ldots\ldots$

Let the scale of relation be denoted by m, n, p, q.

$\therefore 921x^4=180qx^3+35px^2+6nx+m \ldots\ldots\ldots\ldots (1).$

$4626x^5=921qx^4\times 180px^3+35nx^2+6mx \ldots\ldots\ldots\ldots (2).$

$23215x^6=4626qx^5+921px^4+180nx^3+35mx^2 \ldots\ldots\ldots\ldots (3).$

$116160x^7=23215qx^6+4626px^5+921nx^4+180mx^3 \ldots\ldots\ldots\ldots (4).$

$\therefore m=20x^4,\ n=-24x^3,\ p=-x^2,\ q=6x.$

Since the series has a quadruple scale of relation it must be composed of the sum of four geometrical series. The ratios of these series will be the roots of the biquadratic equation

$r^4=6xr^3-x^2r^2-24x^3r+20x^4 \ldots\ldots\ldots\ldots (5).$

$\therefore r_1=2x,\ r_2=-2x,\ r_3=5x,\ r_4=x.$

Let a_1, a_2, a_3, a_4 be the first terms of these sets of series ; then

$a_1+a_2+a_3+a_4=1 \ldots\ldots\ldots\ldots (6).$

$a_1r_1+a_2r_2+a_3r_3+a_4r_4=2a_1-2a_2+5a_3+a_4=6 \ldots\ldots\ldots\ldots (7).$

$a_1r_1{}^2+a_2r_2{}^2+a_3r_3{}^2+a_4r_4{}^2=4a_1+4a_2+25a_3+a_4=35 \ldots\ldots\ldots\ldots (8).$

$a_1r_1{}^3+a_2r_2{}^3+a_3r_3{}^3+a_4r_4{}^3=8a_1-8a_2+125a_3+a_4=180 \ldots\ldots\ldots\ldots (9).$

$\therefore a_1=-2/3,\ a_2=2/21,\ a_3=125/84,\ a_4=1/12.$

Hence the series are :

$-2/3-4x/3-8x^2/3-16x^3/3-32x^4/3-64x^5/3-\ldots\ldots\ldots\ldots (10).$

$2/21-4x/21+8x^2/21-16x^3/21+32x^4/21-64x^5/21+\ldots\ldots\ldots\ldots (11).$

$125/84+625x/84+3125x^2/84+15625x^3/84$

$+78125x^4/84+390625x^5/84+\ldots\ldots\ldots\ldots (12).$

$1/12+x/12+x^2/12+x^3/12+x^4/12+x^5/12+\ldots\ldots\ldots\ldots\ldots(13)$.

Let A_n^1, A_n^2, A_n^3, A_n^4, S_n^1, S_n^2, S_n^3, S_n^4 represent the nth terms, and the sum of n terms of the series (10), (11), (12), (13). Then,

$$A_n^1=-\tfrac{2}{3}(2x)^{n-1},\ A_n^2=\tfrac{2}{21}(\pm 2x)^{n-1},\ A_n^3=\tfrac{125}{84}(5x)^{n-1},\ A_n^4=\tfrac{1}{12}x^{n-1},$$

$$S_n^1=-\tfrac{2}{3}\left(\frac{2^nx^n-1}{2x-1}\right),\ S_n^2=\tfrac{2}{21}\left(\frac{\pm 2^nx^n-1}{-2x-1}\right),\ S_n^3=\tfrac{125}{84}\left(\frac{5^nx^n-1}{5x-1}\right),$$

$$S_n^4=\tfrac{1}{12}\left(\frac{x^n-1}{x-1}\right).$$

Let A_n, S_n, be the nth term and the sum of n terms of the original series.

$$\therefore\ A_n=\tfrac{1}{84}\{5^{n+2}+7(1-2^{n+2})\mp 2^{n+2}\}x^{n-1}.$$

$$S_n=\tfrac{1}{84}\left\{\frac{125(5^nx^n-1)}{5x-1}-\frac{56(2^nx^n-1)}{2x-1}+\frac{8(\pm 2^nx^n-1}{-2x-1}+\frac{7(x^n-1)}{x-1}\right\}.$$

The upper sign to be used when n is even. Now let $x=1$, $n=20$, and we will get the required results for the problem. $A_{20}=28383163779300$, the number the twentieth year; $S_{20}=35478954491110$, the number in twenty years.

Also solved by *EDWARD R. ROBBINS.*

65. Proposed by F. M. SHIELDS, Coopwood, Mississippi.

A, B, and C bought unequal shares in 200 acres of land at the same price per acre, which they sold for \$286.90. A gained as much per cent. on his part as he had acres, B gained 5-8 as much per cent. on his part as A did, and C lost \$9.10 on the cost of his part; the total net gain was 43 9-20 per cent. How much land did each buy, and what did each receive per acre at the sale?

I. Solution by W. H. CARTER, Professor of Mathematics in Centenary College of Louisiana, Jackson, Louisiana.

Let x, y, and z be the number of acres bought by A, B, and C, respectively. $\therefore\ x+y+z=200\ldots\ldots\ldots\ldots\ldots\ldots(1)$.

Since the selling price is \$286.90 and the gain per cent. is 43.45, the cost is \$200. Let m=cost per acre; then mx, my, and mz represent the cost of the shares of A, B, and C, respectively. $\therefore\ m(x+y+z)=200$. $\therefore\ m=1$. $\therefore$ the cost of the share of each=number of acres he bought.

x=A's gain per cent., and $5x/8$=B's gain per cent.

$\therefore\ x+x^2/100+y+5xy/800+z-\$9.10=\$286.90$.

$\therefore\ x^2/100+5xy/800=\$96$. $\therefore\ 8x^2+5xy=76800$.

$$\therefore\ y=\frac{76800-8x^2}{5x}=\frac{15360}{x}-\frac{8x}{5}\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If the number of acres bought by each is to be integral, then (1) and (2) are to be solved for *positive integral* values of x, y, and z. Since y is to be integral, x must be a factor 15360 and must be divisible by 5. $15360=5\times3\times2^{10}$. $\therefore$ the factors of 15360 which are divisible by 5, are 5, 10, 15, 20, 30, 40, 60, 80, 120, 160, etc. If x has any of these values less than 80, z will be negative; if x has any values greater than 80, y is negative. If $x=80$, $y=64$, and $z=56$. $\therefore$ 80, 64, and 56 are the shares of A, B, and C.

The amounts each received per acre at the sale are easily found to be \$1.80, \$1.50, and \$0.83$\frac{3}{4}$.

II. **Solution by EDWARD R. ROBBINS, Master in Mathematics and Physics in the Lawrenceville School, Lawrenceville, New Jersey.**

Let x, y, and $200-x-y$ represent the number of acres which A, B, and C bought, respectively. Then by the problem,

$$x+x^2/100+y+5xy/800+200-x-y-9.10=286.90.$$

This gives $8x^2+5xy=76{,}800$; or $y=(76800-8x^2)/5x$. Solving for positive integers in x, we have, when

$$x=\ 75,\quad 80,\quad 85,\quad 90,$$
$$y=107\tfrac{3}{7},\quad 64,\quad 44\tfrac{12}{17},\quad 26\tfrac{2}{3},$$

Accepting the integral values we obtain:
A's purchase consisted of 80 acres and sold for \$144;
B's purchase consisted of 64 acres and sold for \$96;
C's purchase consisted of 56 acres and sold for \$46.90.
Hence A received \$1$\frac{4}{5}$ per acre; B, \$1$\frac{1}{2}$; and C, \$$\frac{67}{80}$.

III. **Solution by H. C. WILKES, Skull Run, West Virginia.**

Since by the terms of the problem the price paid for the land was \$1 per acre, let $8x$, y, z be the number of acres bought, and the number of dollars paid, by A, B, and C, respectively.

Then $8x+y+z=200\ldots(1)$. $8x+64x^2/100+y+5xy/100+z=296\ldots(2)$.

Subtracting (1) from (2), and clearing, $64x^2+5xy=9600$. Factoring, $x(64x+5y)=10(960)$. Let $x=10$; then $5y=320$, and $y=64$.

$\therefore$ 80, 64, 56 are numbers satisfying the conditions. See solution of a similar problem on page 76 of Vol. II.

IV. **Solution by A. M. HUGHLETT, A. M., Associate Principal and Professor of Mathematics in Randolph-Macon Academy, Bedford City, Virginia.**

Let x, y, and z represent the shares of A, B, and C, respectively. $x+y+z=200\ \ldots..(1)$. Since C lost \$9.10, he must have bought at least 9.10 acres. Therefore 190.90 is the maximum limit of $x+y$.

$$x+x^2/100+y+xy/160+z=296\ldots\ldots(2).$$

$$(1) \text{ in } (2) \text{ gives } x^2/100+xy/160=96.\quad \therefore y=(76800-8x^2)/5x\ldots..(3).$$

$\therefore$ $190.90 > x + (76800 - 8x^2)/5x$. $\therefore$ $190.90 > (76800 - 3x^2)/5x$.
As x decreases, $(76800 - 3x^2)/5x$ increases.

$\therefore$ the equation $(76800 - 3x^2)/5x = 190.90$ .... (4)

gives the minimum limit of x.

$\therefore$ $66.48+$ is the minimum limit of x......... .... (5).

From (3), $y = (76800 - 8x^2)/5x$, we get, since y must have some value, $76800 > 8x^2$; hence $8x^2 = 76800$ gives maximum limit of x. $\therefore$ $97.97+$ is the maximum limit of x. Hence, any values of x between $66.48+$ and $97.97+$ will satisfy the conditions of the problem. *Example:* Let $x = 77\frac{1}{7}$. Then from (3) $y = 75\frac{43}{63}$; $\therefore$ $z = 47\frac{11}{63}$.

$\therefore$ A received \$136.65$\frac{135}{441}$; B received \$112.17$\frac{103}{441}$. $\therefore$ C received \$38.07$\frac{203}{441}$; but he paid \$47.17$\frac{203}{441}$. $\therefore$ C lost \$9.10.

Also solved by *A. H. HOLMES*, *J. SCHEFFER*, and *G. B. M. ZERR*.

PROBLEMS.

72. Proposed by CHAS. C. CROSS, Laytonsville, Maryland.

Prove that $\dfrac{2\sqrt{2+\sqrt{3}}}{4\times\sqrt{6}-\sqrt{2}} = \sqrt{6} - \sqrt{2} + \sqrt{3} - 2$, when reduced to its lowest terms.

73. Proposed by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Find the worth of each of five persons, A, B, C, D, and E, knowing, 1st, that when A's worth is added to a times what B, C, D, and E are worth, it is equal to m; 2nd, when B's worth is added to b times what A, C, D, and E are worth, it is equal to n; 3rd, when C's worth is added to c times what A, B, D, and E are worth, it is equal to p; 4th, when D's worth is added to d times what A, B, C, and E are worth, it is equal to q; 5th, when E's worth is added to e times what A, B, C, and D are worth, it is equal to r.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

51. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the maximum ellipsoid that can be cut out of a given right conic frustum.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

A complete solution of this problem without any assumptions would be a task greater than I care to undertake at present. We will, therefore, assume the cone to be one of revolution. Let $2h$=height of frustum, R, r radii of the lower and upper bases, respectively, l, m, p the coordinates of the vertex.

$x^2/a^2+y^2/b^2+z^2/c^2=1$, the equation to the ellipsoid.

$\therefore (p-z)^2+(n-y)^2=[(R-r)/2h]^2(m-x)^2$ is the equation to the cone.

We will further assume that this cone is the tangent cone to the maximum ellipsoid, then the equation to the cone is

$$(m^2/a^2+n^2/b^2+p^2/c^2-1)(x^2/a^2+y^2/b^2+z^2/c^2-1)$$
$$=(mx/a^2+ny/b^2+pz/c^2-1)^2.$$

From these two equations to the cone we get $n=p=0$.

$[(R-r)/2h]^2=Rr/(m^2-h^2)$ or $m=[(R+r)/(R-r)]h$.

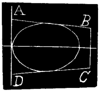
$\therefore$ The center of the frustum and the center of the ellipsoid coincide, and the ellipsoid is one of revolution.

$\therefore x^2/a^2+(y^2+z^2)/b^2=1$ is its equation. $\therefore a=h$, $b=\sqrt{Rr}$.

$V=\frac{4}{3}\pi hRr$=volume of maximum ellipsoid.

II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

The figure shows vertical section of frustum and inscribed ellipsoid, with axis of x coinciding with axis of cone, and axis of y in base. Let d and c be radii of bases, and h the altitude. Then $(0, d)$ and (h, c) represent points A and B, respectively.



$(x-a)^2/a^2+y^2/b^2=1$ is equation to inscribed ellipse. $\therefore$ equation to AB, as tangent to ellipse, is

$$a^2yy_1+b^2(x-a)(x_1-a)=a^2b^2 \quad \ldots\ldots\ldots\ldots (1),$$

x_1 and y_1 being coordinates of point of contact. Substituting coordinates of A and B for x and y in (1),

$$\left.\begin{aligned} a^2dy_1+b^2(-x)(x_1-a)=a^2b^2 \\ a^2cy_1+b^2(h-a)(x_1-a)=a^2b^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2).$$

Solving (2) for x_1 and y_1,

$$\left.\begin{aligned} x_1=ahd/(bd-ad+ac) \\ y_1=b^2h/(bd-ad+ac) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3).$$

Substituting from (3) for x and y in equation of ellipse and solving we obtain $b^2=[(hd^2+2ad(c-d)]/h$.

Now volume of ellipsoid $V=4/3(\pi ab^2)-4\pi/3h[ad^2h+2a^2d(c-d)]$.

$$dV/da=4\pi/3h[d^2h+4ad(c-d)] \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Equating (4) to 0, we find $a=dh/4(d-c)$..........................(5),

and $b^2=d^2/2$............................(6).

Also, $d^2V/da^2=16\pi d(c-d)/3h$, which is negative since $d>c$. Now the ellipsoid will be entirely within the frustum if $2a$ is not greater than h, which from (5) gives, $dh/2(d-c)$ is not greater than h or c is not greater than $\frac{1}{2}d$. So volume of maximum ellipsoid=

$\frac{4\pi}{3}\cdot\frac{dh}{4(d-c)}\cdot\frac{d^2}{2}=\frac{\pi}{6}\frac{d^3h}{d-c}$, if c is not greater than $\frac{1}{2}d$, or $\frac{4\pi}{3}hb^2$, if $c>\frac{1}{2}d$, the latter result being true, since (4) shows but one maximum, and V is a continuous function of A.

III. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Take the base of the frustum as the plane xz, and the axis of the frustum as the axis of y. We may, without loss of generality, take one axis parallel to the axis of z. The equation of the ellipsoid may then be written :

$$Ax^2+By^2+Cxy+Dx+Ey+Hz^2+F=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

We find the axes of the ellipsoid to be:

$$a=\sqrt{R/P},\ b=\sqrt{R/Q},\ c=\sqrt{R/H},$$

where $R=F(c^2-4AB)+AE^2+BD^2-CD^2)/(4AB-C^2)$.

$P=1/2[A+B\pm\sqrt{(A-B)^2+C^2}]$,

$Q=1/2[A+B\mp\sqrt{(A-B)^2+C^2}]$,

Volume of ellipsoid$=4/3(\pi abc)$

$$=\tfrac{8}{3}\pi\frac{[F(C^2-4AB)+AE^2+BD^2-CDE]^{\frac{3}{2}}}{[4AB-C^2]^2}\cdot\frac{1}{\sqrt{H}} \quad \ldots\ldots\ldots\ldots(2).$$

A little consideration will show that the ellipsoid to be a maximum *must touch* the larger base of the frustum and also the conical surface. The condition that it touch the lower base is $D^2-4AF=0$.............................(3).

The condition that it shall not cut the upper base is

$$(Ch+D)^2-4A(Bh^2+Eh+F(<0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where h is the altitude of the frustum.

To find the condition that the ellipsoid shall be tangent to conical surface, we assume the equation of complete cone to be:

$$m^2(x^2+z^2)=(y-k)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

For intersection of (1) and (5),

$$(A-H)m^2x^2+(Bm^2+H)y^2+Cm^2xy+Dm^2x+(Em^2-2k)+(Hk^2+Fm^2)=0.$$

If this ellipse have no axes,

$$(Hk^2+Fm^2)[c^2m^2-4(A-H)(Bm^2+H)]+(A-H)(Em^2-2k)^2+(Bm^2+H)D^2m^2$$
$$-CD(Em^2-2k)m^2=0.$$

Solving this for B we obtain,

$$B=\frac{CD(Em^2-2k)m^2-(A-H)(Em^2-2k)^2-C^2m^2(Hk^2+Fm^2)}{m^2[D^2m^2-4(A-H)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

Substitute the value of A given in (3),

$$B=\frac{4FCD(Em^2-2k)m^2-(D^2-4FH)(Em^2-2k)^2-4FC^2m^2(Hk^2+Fm^2)}{m^2[FD^2m^2-(D^2-4FH)(Hk^2+Fm^2)]}-\frac{H}{m^2}.$$

If we were then to substitute these values of A and B in equation (2), we should obtain a value of V which contains the variables C, D, E, F, and H, independent, except as to the condition given in (4). By the ordinary methods of maximum and minimum, five equations can be formed and the maximum critical values of the five letters determined. But life is too short to do this.

If we assume that two of the axes are parallel to the bases of the frustum, we obtain $V=\frac{4}{3}\pi\tan^2\phi(h^2b-2hb^2)$, where h=altitude of complete cone, ϕ=semi-angle of cone, and b=semi-vertical axis of ellipsoid. From this for maximum, $b=p/4$.

52. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

There are two lights of intensities m and n. Where must a target, whose surface is parallel to the line joining the two lights, be set up in order that it shall receive the maximum illumination per unit of area?

I. Solution by the PROPOSER.

If we take the point where the light with intensity l is situated as the origin of coordinates, we have readily from the principles of Optics,
$I=ly/(x^2+y^2)^{\frac{3}{2}}+my/[(a-x)^2+y^2]^{\frac{3}{2}}$, x and y being the coordinates of the bull's-eye.

$$\frac{dI}{dx}=\frac{-3lxy}{(x^2+y^2)^{\frac{5}{2}}}+\frac{3m(a-x)y}{[(a-x)^2+y^2]^{\frac{5}{2}}}=0 \ldots\ldots \ldots\ldots\ldots (1).$$

$$\frac{dI}{dy}=\frac{l(x^2-2y^2)}{(x^2+y^2)^{\frac{5}{2}}}=\frac{m[(a-x)^2-2y^2]}{[(a-x)^2+y^2]^{\frac{5}{2}}}=0 \ldots\ldots\ldots\ldots\ldots (2).$$

From (1) $y=0$ ....(a),

$$\text{or } \frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{[x^2+y^2]^{\frac{5}{2}}}=\frac{m}{l}\cdot\frac{a-x}{x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

$$\text{From (2) } \frac{[(a-y)^2+y^2]^{\frac{5}{2}}}{[x^2+y^2]^{\frac{5}{2}}}=-\frac{m}{l}\cdot\frac{(a-x)^2-2y^2}{x^2-2y^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$$

From (b) and (c) $y^2=x(a-x)/2$ ....(d).

$$\text{From } (b)\ \ y=\pm x^{\frac{1}{5}}(a-x)^{\frac{1}{5}}\sqrt{\frac{m^{\frac{2}{5}}x^{\frac{7}{5}}-l^{\frac{2}{5}}(a-x)^{\frac{7}{5}}}{l^{\frac{2}{5}}x^{\frac{2}{5}}-m^{\frac{2}{5}}(a-x)^{\frac{2}{5}}}} \ldots\ldots\ldots\ldots\ldots\ \ldots (e).$$

By (a) and (d), $x=0$; $x=a$; that is, the lights themselves must be used as bull's-eye. By (a) and (e) we obtain the additional condition $x=al^{\frac{2}{7}}/(l^{\frac{2}{7}}+m^{\frac{2}{7}})$, which is the point of minimum illumination on the line joining the two lights. Other critical points will be obtained by solving (d) and (e) simultaneously,—a task which seems to be almost impossible.

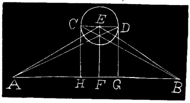
II. Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let A, B, be the lights, intensities m, n; E the center of the target, radius $ED=r$, $AB=a$, $AF=x$, $EF=z$, $\angle DAB=\theta$, $\angle CBA=\phi$.

$\therefore\ m\sin\theta/AD^2+n\sin\phi/BC^2=I$.

$\sin\theta=z/AD=z/\sqrt{z^2+(x+r)^2}$,

$\sin\phi=z/\sqrt{z^2+(a-x+r)^2}$.



$$\therefore \frac{mz}{\{z^2+(x+r)^2\}^{\frac{3}{2}}}+\frac{nz}{\{z^2+(a-x+r)^2\}^{\frac{3}{2}}}=I.$$

Differentiating with reference to z,

$$\frac{m(x+r)^2-2mz^2}{\{z^2+(x+r)^2\}^{\frac{5}{2}}}+\frac{n(a-x+r)^2-2nz^2}{\{z^2+(a-x+r)^2\}^{\frac{5}{2}}}\ \ 0 \ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating with respect to x,

$$\frac{m(x+r)}{\{z^2+(x+r)^2\}^{\frac{5}{2}}}=\frac{n(a-x+r)}{\{z^2+(a-x+r)^2\}^{\frac{5}{2}}} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2), $z=\sqrt{\frac{1}{2}(x+r)(a-x+r)}$. This value of z in (2) gives

$$\frac{m(x+r)}{\{(x+r)(a+x+3r)\}^{\frac{3}{2}}}=\frac{n(a-x+r)}{\{(a-x+r)(2a+3r-x)\}^{\frac{3}{2}}},$$

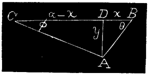
an equation of the eighth degree to find x.

If $m=n$, $x=\frac{1}{2}a$, $z=(\frac{1}{2}a+r)\sqrt{\frac{1}{2}}=\frac{1}{2}(\frac{1}{2}a+r)\sqrt{2}$.

If $n=0$, $x=0$, $z=\frac{1}{2}r\sqrt{2}$.

III. Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Let A be any position of target; $AD(=y)$ be perpendicular from A to BC, the line connecting the positions of the two lights. Let x equal part of $BC(=a)$ cut off by AD. By laws of light, intensity of light received from B at A



$$=\frac{m\sin\theta}{AB^2}=\frac{m}{x^2+y^2}\times\frac{y}{\sqrt{x^2+y^2}}=\frac{my}{(x^2+y^2)^{\frac{3}{2}}}.$$

Similarly, that received from $C=\dfrac{ny}{[(a-x)^2+y^2]^{\frac{3}{2}}}$.

Then total intensity at A or $u=my(x^2+y^2)^{-\frac{3}{2}}+ny[(a-x)^2+y^2]^{-\frac{3}{2}}$. ..(1).

$$du/dx=-3mxy(x^2+y^2)^{-\frac{5}{2}}+3n(a-x)y[(a-x)^2+y^2]^{-\frac{5}{2}}\ldots\ldots\ldots.(2).$$

$$du/dy=m(x^2+y^2)^{-\frac{3}{2}}-3my^2(x^2+y^2)^{-\frac{5}{2}}+n[(a-x)^2+y^2]^{-\frac{3}{2}}-3ny^2[(a-x)^2+y^2]^{-\frac{5}{2}}\ldots\ldots\ldots(3).$$

Equating (3) to 0, we have

$$\left.\begin{array}{l} y=0\ldots\ldots\ldots\ldots\ldots\ldots(4). \\ \dfrac{[(a-x)^2+y^2]^{\frac{5}{2}}}{(x^2+y^2)^{\frac{5}{2}}}=\dfrac{n(a-x)}{mx}\ldots\ldots\ldots(5). \end{array}\right\}$$

Equating (3) to 0, we have

$$[(a-x)^2+y^2]^{-\frac{5}{2}}\{3ny^2-n[(a-x)^2+y^2]\}=(x^2+y^2)^{-\frac{5}{2}}[m(x^2+y^2)-3my^2],$$

$$\text{or }\frac{[(a-x)^2+y^2]^{\frac{5}{2}}}{(x^2+y^2)^{\frac{5}{2}}}=\frac{n[2y^2-(a-x)^2]}{m(x^2-2y^2)}\ldots\ldots\ldots\ldots(6)$$

Solving (4) and (6), $\dfrac{(a-x)^5}{x^5}=\dfrac{-n(a-x)^2}{mx^2}$, which gives

$$\left\{\begin{array}{l} x=a \text{ or } 0 \text{ or } \dfrac{a}{1-\sqrt[3]{m\div n}} \\ \text{and } y=0, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

From (5) and (6) $\dfrac{n(a-x)}{mx}=\dfrac{n[2y^2-(a-x)^2]}{m(x^2-2y^2)}$, and $y^2=\dfrac{x(a-x)}{2}$......(8)

Instead of finding second differential coefficients, substitute from (7) in (1), $x=a$ and $x=0$, make $n=\infty$. $x=\dfrac{a}{1-\sqrt[3]{n\div m}}$, makes $n=0$.

We can show that (8) does not produce any new condition for a maximum. To make y real x is not <0 nor $>a$.

If $x=a$ or 0, we have the values found in (7). Now for any value of x between 0 and a, y in (8) is seen to be finite, and n in (1) is also finite.

So $x=a$ or $x=0$ with $y=0$, producing the only infinite values of n indicate the positions of maximum intensity of illumination to be directly in front of either light.

PROBLEMS.

59. Proposed by MOSES C. STEVENS, M. A., Department of Mathematics, Purdue University, Lafayette, Indiana.

Solve $n\dfrac{d^2y}{dx^2}(x^2+y^2)^{\frac{1}{2}}=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$.

[*From Forsyth's Differential Equations.*]

60. Proposed by SETH PRATT, C. E., Assyria, Michigan.

To remove $(1/a)$th of the volume of a sphere of a given radius by a conical hole, whose axis is the axis of a sphere, and whose vertex is at the surface of the sphere. Required the height of the cone and the diameter of its base.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

NOTE ON PROBLEM 26.

After carefully reading Dr. Martin's "Reply to Replies on Problem 26," we see no reason for changing our opinion respecting the solution we have been defending. We may, however, be led to agree with Dr. E. H. Moore, Dr. William

Hoover, and Prof. Henry Heaton, that there is no *correct* solution of the problem. That is to say, in so much as the problem is stated in the indefinite form, a solution taking any one of the elements of a triangle of which the area is a function will lead to a correct result. But it does seem to us that this statement, while it is true in general, does not apply in this case. It was asked of us during the summer whether anyone had sent in a solution assuming the altitude as the variable. Now we think it is quite clear that a solution which assumes the altitude of the triangle as the variable can, in no way be correct, for the solution would include not only right triangles, but oblique triangles as well. The result is

$$2\int_0^{\frac{1}{2}a}\tfrac{1}{2}apdp\div\int_0^{\frac{1}{2}a}dp=\tfrac{1}{4}a^2,\text{ where } p \text{ is the altitude.}$$

But if p is made a function of the angle at the center of the circle subtended by a side, the result will be $\frac{a^2}{2\pi}$. We think, however, that this controversy has been carried on long enough, and therefore it is desirable that it close without further discussion. EDITOR.

NOTE ON PROBLEM 29.

BY HENRY HEATON.

When the points to which r is measured are distributed symmetrically with respect to the minor axis the different radii vectores may be arranged in pairs such that the sum of the lengths of each pair will be $2a$. Hence, using Dr. Hoover's notation, m'' and m''' each equal a. m''' may be shown to equal a by the calculus, thus :

$$r=a-ex,\ d\theta=\frac{(a^2-e^2a^2)^{\frac{1}{2}}dx}{(a^2-x^2)^{\frac{1}{2}}}.$$

Hence $$m'''=\int_{-a}^{+a}\frac{(a-ex)(a^2-e^2x^2)^{\frac{1}{2}}dx}{(a^2-x^2)^{\frac{1}{2}}}\div\int_{-a}^{+a}\frac{(a^2-e^2x^2)^{\frac{1}{2}}dx}{(a^2-x^2)^{\frac{1}{2}}}$$

$$=a\int_{-a}^{+a}\frac{(a^2-e^2x^2)^{\frac{1}{2}}dx}{(a^2-x^2)^{\frac{1}{2}}}\div\int_{-a}^{+a}\frac{(a^2-e^2x^2)^{\frac{1}{2}}dx}{a^2-x^2}=a.$$

A fourth very obvious case of this problem is when the distances are measured at equal intervals of time.

Then $$m''''=\int rdA\div\int dA=\int_0^{\pi}\frac{r^3}{2}d\theta\div\int_0^{\pi}\frac{r^2}{2}d\theta$$

$$=\frac{a^2(1-e^2)^3}{b\pi}\int_0^{\pi}\frac{d\theta}{(1-e\cos\theta)^3}=\frac{a^2}{2b}\left(3(4+e^2)(1-e^2)^{\frac{1}{2}}-10(1-e^2)^2\right).$$

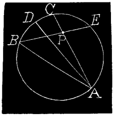
Corollary. Let $e=0$, then $m''''=a$, as it evidently should.

32. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of the random sector whose vertex is a random point in a given circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let P be the random point. Through P draw the chords AC, DE forming the sector DPC. From A draw the diameter AB and the chord AD. Let $AB=2r$, $AP=z$, $\angle BAP=\varphi$, $\angle BAD=\theta$, area $DPC=u$, $\Delta=$required average. Then



$$u=r^2(\varphi-\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{2}\sin 2\varphi)-rz\cos\theta\sin(\varphi-\theta).$$

The limits of θ are $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of φ, θ and $\frac{1}{2}\pi$; of z, 0 and $2r\cos\varphi=z'$.

$$\therefore\ \Delta=\frac{\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_{0}^{z'}u\,d\theta\,d\varphi\,z\,dz}{\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_{0}^{z'}d\theta\,d\varphi\,z\,dz}=\frac{2}{\pi^2r^2}\int_{\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}\int_{0}^{z'}u\,d\theta\,d\varphi\,z\,dz,$$

$$=\frac{2r^2}{3\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_{\theta}^{\frac{1}{2}\pi}[3\cos^2\varphi(2\varphi-2\theta-\sin 2\theta+\sin 2\varphi)-8\cos\theta\cos^3\varphi\sin(\varphi-\theta)]\,d\theta\,d\varphi$$

$$=\frac{r^2}{12\pi^2}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}(3\pi^2-12\pi\theta+\theta^2-16\cos^2\theta+4\sin^2\theta\cos^2\theta)\,d\theta=\frac{37\pi r^2}{144}-\frac{5r^2}{8\pi}.$$

Also solved by the *PROPOSER*.

33. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all regular polygons having a constant apothem.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let $a=$constant apothem, $2x=$side, $2\theta=$central angle of polygon.

$\therefore\ \dfrac{\pi}{\theta}=$number of sides, $\dfrac{\pi ax}{\theta}=$area of polygon.

$$\therefore\ A=\text{average area}=\pi a\frac{\int_{0}^{a\sqrt{3}}\frac{x\,dx}{\theta}}{\int_{0}^{a\sqrt{3}}dx}=\frac{\pi}{\sqrt{3}}\int_{0}^{a\sqrt{3}}\frac{x\,dx}{\theta}$$

$$=\frac{\pi a^2}{\sqrt{3}}\int_{0}^{\frac{1}{3}\pi}\frac{\tan\theta\sec^2\theta\,d\theta}{\theta}\quad\text{where } x=a\tan\theta$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi a^2}{2\sqrt{3}}\int_0^{\frac{1}{3}\pi}\left(\frac{\tan\theta}{\theta}\right)^2 d\theta$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi a^2}{2\sqrt{3}}\int_0^{\frac{1}{3}\pi}(1+\tfrac{2}{3}\theta^2+\tfrac{17}{45}\theta^4+\tfrac{62}{315}\theta^6+\ldots\ldots)d\theta,$$

$$=\frac{3\sqrt{3}a^2}{2}+\frac{\pi^2a^2}{6\sqrt{3}}\left(1+\frac{2\pi^2}{81}+\frac{17\pi^4}{18225}+\frac{62\pi^6}{1607445}+\ldots\ldots\right)=3.8693a^2 \text{ nearly.}$$

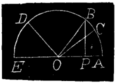
Also solved by the *PROPOSER*.

34. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Two points are taken at random on the circumference of a semicircle. Find the chance that their ordinates fall on either side of a point taken at randon on the diameter.

Solution by G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

Let P be the random point on the diameter AE. Draw BP perpendicular to AE. Then one point must fall somewhere, as at C, on arc AB, the other somewhere, as at D, on arc BE. The chance thus obtained must be doubled as D might fall on AB and C on BE.



Let AO=unity, $\angle BOA=\theta$, $\angle COA=\varphi$, $\angle DOA=\psi$.

Then $OP=\cos\theta$. $\therefore d(OP)=-\sin\theta d\theta$.

Let p=required chance.

$$\text{Then } p=\frac{\int_0^\pi\int_0^\theta\int_\theta^\pi \sin\theta d\theta d\varphi d\psi}{\int_0^\pi\int_0^\pi\int_0^\pi \sin\theta d\theta d\varphi d\psi}=\frac{1}{\pi^2}\int_0^\pi\int_0^\theta\int_\theta^\pi \sin\theta d\theta d\varphi d\psi$$

$$=\frac{1}{\pi^2}\int_0^\pi(\pi\theta-\theta^2)\sin\theta d\theta=\frac{4}{\pi^2}.$$

PROBLEMS.

42. Proposed by CHARLES E. MYERS, Canton, Ohio.

A attends church 4 Sundays out of 5; B, 5 Sundays out of 6; and C, 6 Sundays out of 7. What is the probability of an event that A and B will be at church and C will not?

43. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

In a circle whose radius is a, chords are drawn through a point distant b from the center. What is the average length of such chords, (1), if a chord is drawn from every point of the circumference, and (2), if they are drawn through the point at equal angular intervals ?

EDITORIALS.

Prof. John N. Lyle, of Westminster College, has resigned his position on account of ill health, and is now living in Bentonville, Arkansas.

We shall be pleased to have our subscribers send us the names of persons likely to subscribe for the MONTHLY, in order that we may send such persons sample copies.

Any reader of the MONTHLY having a copy of *Salmon's Higher Plane Curves*, third edition, and wishing to sell the same, should write to us stating the price of the book.

We have only six complete sets of Volumes I and II, of the MONTHLY. Volume I will be sent to any address in the United States or Canada for $2.00; Volume II will be sent on receipt of $2.50.

Prof. Robert J. Aley, of Indiana University, is now studying mathematics in the University of Pennsylvania, having received a Mathematical Fellowship in that Institution last spring.

In our August-September number, we sent out bills to all those who are owing us. We hope that the matter of remittance may receive the attention of all those who are in arrears, as the MONTHLY is greatly in need of funds. All bills not paid by December 31st will be sent to an attorney for collection.

Prof. A. B. Nelson, of Centre College, Kentucky, says, in a letter of October 13th, "You deserve the thanks of mathematicians in this country for your self-sacrificing labors in behalf of our favorite science." We desire to thank Professor Nelson as well as many others who have thus expressed their appreciation of our labor. Surely it is a labor of love.

BOOKS AND PERIODICALS.

Elementary Solid Geometry and Mensuration. By Henry Dallas Thompson, D. Sc., Ph. D., Professor of Mathematics in Princeton University. 8vo. Cloth, 200 pages. Price, $1.25. New York: The Macmillan Co.

In this book, the author lays the foundation of his subject in clear cut and accurate *definitions* and well illustrated *postulates*. The diagrams are very fine, showing very accurately to the eye the relation of the points, lines, and planes. There are numerous original exercises scattered throughout the book. B. F. F.

The Elements of Algebra. Adapted for use in High Schools, Academies, and Colleges. By Lyman Hall, Graduate United States Military Academy, and Professor of Mathematics, Georgia School of Technology. 8vo. Cloth and Leather Back, 368 pages. Chicago: American Book Co.

This work is intended for beginners who have mastered the principles of any good common school Arithmetic. The familiar methods of arithmetic are preserved, in order to gradually convince the student that algebra is merely an extension of the mathematical knowledge he already possesses. *Preface.* B. F. F.

Trigonometry for Beginners. By the Rev. J. B. Lock, M. A., Fellow of Gonville and Caius College, Cambridge, Formerly Master at Eaton. Revised and Enlarged for the use of American Schools, by John A. Miller, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Professor (elect) of Mechanics and Mathematical Astronomy, Indiana University. Large 8vo. Cloth, 148 and 64 pages. Price, \$1.10.

As we have not seen the original book, we do not know just how materially Professor Miller has changed it. He tells us in his Preface that it differs from the original, chiefly in the following particulars: (1) The subject matter of Chapter VII formerly followed that of Chapters VIII and IX; (2) the addition formulæ are proved for angles of any magnitude, and for more than two angles; (3) a chapter on Inverse Trigonometric Functions; and two chapters on Spherical Trigonometry have been added; (4) logarithmic and trigonometric tables have been inserted. Some of the trigonometrical formulæ are very neatly established by Geometrical Proof. B. F. F.

A School Algebra. Designed for use in High Schools and Academies. By Emerson E. White, A. M., LL. D., Author of "Series of Mathematics," "Elements of Pedagogy," "School Management," etc. 8vo. Cloth and Leather Back, 394 pages. Chicago: American Book Co.

The author's aim has been to prepare a school algebra that is pedagogically sound as well as mathematically accurate. Few educators will question Dr. White's ability to write a work pedagogically sound, but many mathematicians, upon examination of his treatment of *Undetermined Coefficients*, Chapter XXI., will question the mathematical accuracy of his text on algebra. His treatment of Undetermined Coefficients is that given in most algebras written during the last and present century. This demonstration is now pretty generally admitted to be incorrect, and correct demonstrations are being published in most recent works. However, upon the whole, the book is one well suited for the purpose for which it is written. B. F. F.

Elements of Geometry. By Andrew W. Phillips, Ph. D., and Irving Fisher, Ph. D., Professors in Yale University. Large 8vo. Cloth and Leather Back, 540 pages. Price, \$1.75. New York: Harper & Bros.

There are several features in this work that make it especially interesting. Of these the most prominent are the beautiful diagrams. These are photo-engravings arranged side by side with skeleton drawings of geometrical figures. The photographs were taken from actual models recently constructed for use in the class-rooms of Yale University. In this respect the work excels anything that has yet appeared in this country. The work is characterized by clearness of presentation, both in the form of the diagrams and the natural and symmetrical methods of proof. The book closes with a short but very clear treatment of Modern Geometry. This will be helpful to those teachers who desire a knowledge of

the three kinds of Geometries. We believe that this work is destined to be very extensively used throughout the country. B. F. F.

A History of Elementary Mathematics, with Hints and Methods of Teaching. By Florian Cajori, Ph. D., Professor of Physics in Colorado College. 8vo. Cloth, 304 pages. Price, $1.50. New York: The Macmillan Co.

The book is by no means an abridged edition of the author's *History of Mathematics*. It is an entirely new book giving a somewhat detailed account of the rise, struggle, and progress of Arithmetic, Algebra, and Geometry. The book should be read by all teachers of these subjects, and by mathematical students generally. B. F. F.

The Cosmopolitan. An Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year. Single numbes, 10 cents. Irvington-on-the-Hudson, New York.

The October number contains the following: A Summer Tour in the Scottish Highlands; The Story of a Child Trainer; The Perils and Wonders of a True Desert; A Modern Fairy Tale; Hofman's Object Lesson; Personal Recollections of the Tai-Ping Rebellion; The Modern Woman Out-of-Doors; The True History of our Cooks; To a Hyacinth Bulb (poem). B. F. F.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year. Single number, 25 cents. The Review of Reviews Co., New York.

In the September and October numbers of *The Review of Reviews*, the editor has given a remarkably fair and unprejudiced account of the progress of the present political campaign. It is a great satisfaction, after having read statements in the daily papers, which are believed to be misrepresenting, to go to *The Review of Reviews* and get the facts there given by its able editor. The November number contains a very able article on the "Summing Up of the Vital Issues of 1896," by Rev. Dr. Lyman Abbott. Also the question "Would Free Coinage Benefit Wage Earners?" is debated by Dr. Chas. B. Spahr and Prof. Richmond Mayo-Smith. This number also offers a remarkable symposium of current thought on "What Should be Done with Turkey?" The Monthly suggests in answer to this question, that Turkey be given a material and substantial roast by the civilized world.

B. F. F.

Errata.

After the word, ellipses, page 181, problem 60, insert, "passing through the foci of a given ellipse and having the tangents at the ends of the major axes for directrices."

Page 205, line 1, for "$\frac{2}{3}$" read $\frac{3}{2}$.

Page 205, line 1, for "$\frac{3}{2}$" read $\frac{2}{3}$.

Page 205, line 12, for "$4a^3$" read $2a^3$.

Page 206, line 3, for "$(5x^{30}$" read $(5x)^{30}$.

Page 217, line 15, for "y" read z.

Throughout the solution to problem 34, Mechanics, for "E_o^π" read F_o^π.